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Failure criteria for multiply flawed anisotropic materials

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This study presents overall failure criteria for an infinite anisotropic solid containing multiple flaws subjected to a set of uniform applied loads. Based on the inclusion method, flaws are treated as elliptical inclusions where their elastic moduli are considered to be zero. The explicit expression of elastic fields is obtained for a cubic crystal multiply flawed solid through the use of the Mori-Tanaka mean field theory. The resulting expression is further utilized to find an interaction energy function between the applied loads and flaws. With this energy function, the energy release rates and critical stresses are acquired separately in a closed form for Mode I, II, and III. The closed forms for energy release rates and critical stresses reveal that they are a function of the aspect ratio and the volume fraction of flaws, the modes of the loading, and the material properties. As an illustrated numerical example, the energy release rates and the critical stresses that vary with both the aspect ratio and the volume fraction of the flaws in a cubic crystal material are discussed.

1. Introduction

Historically, researches in flawed materials have been focused on the analyses of the materials containing a single flaw. However, it has been evident that most materials contain not a single flaw but multiple flaws, and often the flaw density is very high in some materials. Therefore, the related investigations about the averaging elastic response and overall fracture criterion of multiply flawed materials are needed to obtain the effective fracture criteria.

Many researchers have developed the fracture criteria for single crack problems. Two-dimensional criterion is originally proposed by Griffith [1]. Followed the Griffith's work, Sack [2] and Sneddon [3] had obtained the criteria for penny shaped cracked body for three-dimensional problems. Kassir and Sih [4] had investigated critical stresses and surface energy of flat ellipsoidal cracked materials. The formal derivation based on micromechanics for the single crack problem has been given by Willis [5], Barnett and Asaro [6], Mura and Lin [7], Mura and Cheng [8], Huang and Liu [9] among others, further efforts, however, must be expended to perform a failure study of a solid containing multiple cracks or flaws subjected to applied loading in mode I, II, and III. Accomplishing such a task would allow us to fully exploit the advantages of materials. Therefore, in this study, we study the closed form of the energy release rates and the critical stresses for elliptical flaws involved in an infinite solid subjected to one of three kinds of applied loading.

The objective of this work is to develop an analytical and simple approach for determining the failure criterion for many flaws in a three-dimensional, infinitely extended, anisotropic medium. To this end, this article has focused primarily on the following issues. First, the inclusion method [10] is developed to investigate the elastic fields around an elliptical inclusion in a three-dimensional anisotropic solid. Secondly, the results are extended to the multiply flawed problem by means of the equivalent inclusion method [11]. By using the Mori-Tanaka [12] mean field theory and taking the elastic moduli of the inclusion as zero, explicit solutions for equivalent eigenstrains [10] (or stress-free transformation strains [11]) are obtained for three loading modes: a uniaxial tension, an in-plane shear, and an out-plane shear. Then, the energy release rates and the critical stresses of the Griffith fracture criterion are presented in closed forms for multiply flawed materials subjected to these three loading modes separately. Finally, as an illustrated numerical example, the energy release rates and the critical stresses vary with both aspect ratio and the volume fraction of the flaws in a cubic crystal medium are discussed.

2. The inclusion method

Consider an infinitely extended solid D containing an ellipsoidal inclusion Ω whose elastic moduli C_{ijmn} are the same as the matrix. Here the shape of inclusion is taken as ellipsoid that is capable of treating composite reinforcement geometry ranging from the thin flake to continuous fiber reinforcement. Let ε^*_{ab} be eigenstrain (or stress-free transformation strain) in the inclusion Ω, and zero in the matrix $D-\Omega$. When the eigenstrain in the inclusion are uniform, the induced strain ε_{mn} in Ω can be expressed as

$$\varepsilon_{mn} = S_{mnab}\varepsilon^*_{ab}, \tag{1}$$

where S_{mnab} is the well-known Eshelby tensor [11] for elastic ellipsoidal inclusion problems. In this work, a flaw will be modeled as an elliptical ($a_3 \to \infty$, $a_2/a_1 = a$) inclusion oriented with its generatrix parallel to x_3 axis. Components of the Eshelby tensors for elliptical inclusions in a cubic crystal material are given as [13]:

$$\begin{aligned}
S_{1111} &= \frac{(2+3a)C_{11} + aC_{12}}{2(1+a)^2C_{11}},\\
S_{2222} &= \frac{(3a+2a^2)C_{11} + aC_{12}}{2(1+a)^2C_{11}},\\
S_{1122} &= \frac{-aC_{11} + (2+a)C_{12}}{2(1+a)^2C_{11}},\\
S_{2211} &= \frac{-aC_{11} + (a+2a^2)C_{12}}{2(1+a)^2C_{11}},\\
S_{1133} &= \frac{C_{12}}{(1+a)C_{11}}, \quad S_{2233} = \frac{aC_{12}}{(1+a)C_{11}}, \\
S_{1212} &= S_{1221} = S_{2112} = S_{2121}\\
&= \frac{(1+a+a^2)C_{11} - aC_{12}}{2(1+a)^2C_{11}},\\
S_{1313} &= S_{1331} = S_{3113} = S_{3131} = \frac{1}{2(1+a)},\\
S_{2323} &= S_{2332} = S_{3223} = S_{3232} = \frac{a}{2(1+a)}.
\end{aligned} \tag{2}$$

Then, the corresponding stress inside the inclusion can be obtained as

$$\sigma_{ij} = C_{ijmn}(S_{mnab} - I_{mnab})\varepsilon^*_{ab}, \tag{3}$$

where I_{mnab} represents the fourth order identity tensors, i.e.,

$$I_{mnab} = \frac{1}{2}(\delta_{ma}\delta_{nb} + \delta_{mb}\delta_{na}) \tag{4}$$

with δ_{ma} being the Kronecker's delta

3. Overall elastic fields

So far it has been assumed that both the matrix and the inclusions have the same elastic constants. Now turn to the case of an ellipsoidal inhomogeneity, where matrix and inclusions have different elastic constants. To deal with such a composite, the equivalent inclusion method of Eshelby incorporated the Mori-Tanaka mean field theory will be employed to find the overall elastic fields of the composite. Consider a sufficiently large two-phase composite contains same shaped, randomly distributed, and oriented ellipsoidal inhomogeneities $\Omega = (\Omega_1 + \Omega_2 + \cdots + \Omega_N)$ with elastic moduli C^*_{ijmn} and volume fraction f. The surrounding matrix is denoted by $D-\Omega$ and has elastic constants C_{ijmn}. Let the composite be subjected to a far-field stress σ^0_{ij} on the boundary. In the absence of the inhomogeneities, the strain ε^*_{mn} distributes uniformly. The existence of the inhomogeneity Ω_k provides disturbance in local fields of both the matrix and the kth inhomogeneity. The averages of these quantities are expressed by $\langle\sigma^m_{ij}\rangle$ and $\langle\sigma^\Omega_{ij}\rangle$ for the matrix and the kth inhomogeneity respectively. Hereafter the curly bracket $\langle\,\rangle$ over a field variable denotes its value obtained by volume averaging over the entire composite domain D, and the superscripts 'm' and 'Ω' denote quantities in the matrix and the inhomogeneity respectively.

Since the volume average of the disturbance portion of the stress vanishes, i.e.,

$$\int_D \sigma_{ij}\,\mathrm{d}x = 0, \tag{5}$$

we have

$$(1-f)\langle\sigma^m_{ij}\rangle + f\langle\sigma^\Omega_{ij}\rangle = 0. \tag{6}$$

The average disturbed stress in the matrix and the kth inhomogeneity can respectively be written as

$$\langle\sigma^m_{ij}\rangle = C_{ijmn}\langle\varepsilon^m_{mn}\rangle \quad \text{in } D-\Omega, \tag{7}$$

$$\langle\sigma^{\Omega_k}_{ij}\rangle = C^*_{ijmn}\big(\langle\varepsilon^m_{mn}\rangle + \langle\varepsilon_{mn}\rangle\big) \quad \text{in } \Omega_k, \tag{8}$$

where $\langle\varepsilon^m_{mn}\rangle$ is the average strain, $\langle\varepsilon_{mn}\rangle$ the average disturbance of the otherwise uniform strain in Ω_k. Since all inhomogeneities are of the same shape with the same material properties, the average value over is identical with that over Ω, namely, $\langle\sigma^{\Omega_k}_{ij}\rangle = \langle\sigma^\Omega_{ij}\rangle$.

When the composite is subjected to the uniform far-field applied load σ^0_{ij}, the average stress in the inhomogeneities can be expressed as

$$\sigma^0_{ij} + \langle\sigma^\Omega_{ij}\rangle = C^*_{ijmn}\big(\varepsilon^0_{mn} + \langle\varepsilon^m_{mn}\rangle + \varepsilon_{mn}\big), \tag{9}$$

In the derivation of the equation above, $\langle\varepsilon_{mn}\rangle = \varepsilon_{mn}$ in Ω has been used since the applied load is uniform and the inhomogeneity is ellipsoidal [10].

By means of the equivalent inclusion method [11], the stress in the inhomogeneity can be simulated by those in an equivalent inclusion with the elastic moduli of the matrix and a fictitious eigenstrain ε^*_{mn}, which will be determined in the subsequent development. Therefore, Equation 9 can be written as

$$\begin{aligned}
\sigma^0_{ij} + \langle\sigma^\Omega_{ij}\rangle &= C^*_{ijmn}\big(\varepsilon^0_{mn} + \langle\varepsilon^m_{mn}\rangle + \varepsilon_{mn}\big)\\
&= C_{ijmn}\big(\varepsilon^0_{mn} + \langle\varepsilon^m_{mn}\rangle + \varepsilon_{mn} - \varepsilon^*_{mn}\big),
\end{aligned} \tag{10}$$

where the disturbed field ε_{mn} can be related to the fictitious eigenfield ε^*_{mn} as shown Equation 1.

Then, the average disturbance of stress in the inhomogeneity can be written by substituting Equation 1 into Equation 8 as

$$\langle\sigma^\Omega_{ij}\rangle = C_{ijmn}\langle\varepsilon^m_{mn}\rangle + L_{ijmn}(S_{mnab} - I_{mnab})\varepsilon^*_{ab}, \tag{11}$$

Combining Equations 6, 8 and 11 leads to

$$\langle\varepsilon^m_{mn}\rangle = -f(S_{mnab} - I_{mnab})\varepsilon^*_{ab}. \tag{12}$$

Substitution of Equation 12 into Equation 7 and Equation 9, respectively, yields

$$\langle \sigma_{ij}^{m} \rangle = -f C_{ijmn}(S_{mnab} - I_{mnab})\varepsilon_{ab}^{*}, \quad (13)$$

$$\langle \sigma_{ij}^{\Omega} \rangle = (1-f) C_{ijmn}(S_{mnab} - I_{mnab})\varepsilon_{ab}^{*}. \quad (14)$$

The equivalent eigenstrain ε_{ab}^{*} serves as the cornerstone in this work as it is the key ingredient necessary for the solution of flaw problems. Substituting Equation 12 into the equivalency Equation 10 as can solve it

$$\varepsilon_{ab}^{*} = -U_{abij}^{-1}\left(C_{ijmn}^{*} - C_{ijmn}\right)\varepsilon_{mn}^{0}, \quad (15)$$

where U_{abij}^{-1} is the inverse of U_{ijab} given by

$$U_{ijab} = \left(C_{ijmn}^{*} - C_{ijmnn}\right)S_{mnab} + C_{ijab}. \quad (16)$$

By inspection of the Equation 15 it is seen that to solve the equivalent eigenstrain analytically, the inverse of the fourth-order tensor has to be carried out before proceeding any further. It is noted that, in mapping a tensor into a matrix through the Voigt two-index notation, care should be taken in accounting for the shear strain terms, i.e., the factor of two. Thus, the inversion of a fourth-order tensor is not a standard matrix manipulation. A special scheme for the fourth-order inversion must be developed which is briefly outlined here. First, with the notation, a 6 × 6 matrix is constructed for the given fourth-order tensor. The matrix element in columns 4 to 6 is two times their corresponding tensor component. The 6 × 6 matrix is inverted and is then used to map the corresponding inverse tensor. In mapping the matrix back to the corresponding tensor, each element in columns 4 to 6 is divided by 2 to obtain the tensor element. With this scheme, U_{abij}^{-1} in Equation 16 can be evaluated, followed by the results of the equivalent eigenfields ε_{ab}^{*} in Equation 15 for a flaw in a cubic crystal solid acted on by a set of uniform applied loads. Suppose a flaw is considered the elliptical inhomogeneity where its elastic moduli vanish, complete explicit expressions for the equivalent eigenstrain have been obtained and tabulated below.

For the in-plane shear stress σ_{21}^{0} applied to the multiply flawed solid only:

$$\varepsilon_{12}^{*} = \frac{(1+a)^2 C_{11}\sigma_{21}^{0}}{a(1-f)\left(C_{11}^2 - C_{12}^2\right)} \quad (17)$$

For the tensile stress σ_{22}^{0} applied to the multiply flawed solid only:

$$\varepsilon_{11}^{*} = \frac{\left[C_{11}^2(C_{11}+C_{12}) - (3C_{11}-C_{12})C_{12}^2\right]\sigma_{22}^{0}}{(1-f)\left(C_{11}^2 - C_{12}^2\right)\left[2C_{12}^2 - (C_{11}+C_{12})C_{11}\right]}, \quad (18)$$

$$\varepsilon_{22}^{*} = \frac{\left[(C_{11} + 4aC_{11} + C_{12})C_{12}^2 - (1+2a)(C_{11}+C_{12})C_{11}^2\right]\sigma_{22}^{0}}{(1-f)\left(C_{11}^2 - C_{12}^2\right)\left[2C_{12}^2 + (C_{11}+C_{12})C_{11}\right]}, \quad (19)$$

$$\varepsilon_{33}^{*} = \frac{C_{12}\sigma_{22}^{0}}{(1-f)\left[2C_{12}^2 - (C_{11}+C_{12})C_{11}\right]}. \quad (20)$$

For the anti-plane shear stress σ_{23}^{0} applied to the multiply flawed solid:

$$\varepsilon_{23}^{*} = \frac{(1+a)\sigma_{23}^{0}}{2(1-f)C_{44}}. \quad (21)$$

The overall strain field, denoted by $\langle \varepsilon_{mn}^{C} \rangle$, of the composite can then be obtained as the weighted average of that over each phase:

$$\langle \varepsilon_{mn}^{C} \rangle = \frac{1}{V}\left[\int_{D-\Omega}\left(\varepsilon_{mn}^{0} + \langle \varepsilon_{mn}^{m} \rangle\right)\mathrm{d}x + \int_{\Omega}\left(\varepsilon_{mn}^{0} + \langle \varepsilon_{mn}^{\Omega} \rangle\right)\mathrm{d}x\right]. \quad (22)$$

where V denotes the volume of the entire composite. Substituting Equations 14 and 15 into Equation 22 results in the following equation:

$$\langle \varepsilon_{mn}^{0} \rangle = C_{mnij}^{-1}\sigma_{ij}^{0} + f\varepsilon_{mn}^{*} \quad (23)$$

Then, the corresponding overall stress of the composite are readily derived as

$$\langle \sigma_{ij}^{C} \rangle = \sigma_{ij}^{0} - fC_{ijmn}\varepsilon_{mn}^{*} \quad (24)$$

4. Energy release rate

To determine the flaw extension force, a calculation must be made of the change of total potential energy when the flaw is extended by the amount Δa_1. When the far-field surface traction $\sigma_{ij}^{0}n_i$ is applied on the boundary of the multiply flawed material, the energy release rate G is defined as the change of the potential energy of the material, ΔW. For N infinitesimal flaws to grow, the energy release rate per unit thickness is defined as

$$G = -\frac{\partial}{\partial a_1}\sum_{i=1}^{N}(\Delta W)_i \quad (25)$$

where

$$\Delta W = \frac{1}{2}\int_{D}\left(\sigma_{ij}^{0} + \sigma_{ij}\right)\left(u_{j,i}^{0} + u_{j,i}\right)\mathrm{d}D - \int_{|D|}\left(\sigma_{ij}^{0}n_i\right)\left(u_j^{0} + u_j\right)\mathrm{d}S - \left[\frac{1}{2}\int_{D}\sigma_{ij}^{0}u_{j,i}^{0}\,\mathrm{d}D - \int_{|D|}\left(\sigma_{ij}^{0}n_i\right)u_j^{0}\,\mathrm{d}S\right], \quad (26)$$

with $|D|$ denoting the boundary of the flawed material D. It is observed that ΔW represents the interaction energy between the loading and flaw extension forces.

The energy release rate as defined in Equation 25 is particular convenient for calculation since the interaction energy can be expressed only in term of the applied loads and the equivalent eigenstrain ε^*_{ji} as [10]

$$\Delta W = -\frac{1}{2}\int_\Omega \sigma^0_{ij}\varepsilon^*_{ji}\,\mathrm{d}x = -\frac{1}{2}(2\pi a_1 a_2)\sigma^0_{ij}\varepsilon^*_{ji}, \quad (27)$$

where $2\pi a_1 a_2$ is the volume per unit thickness of an elliptical flaw.

Substituting ε^*_{ji} listed in Equations 17–21 into Equation 25 with Equation 27 results in three modes of energy release rate explicitly. For Mode I:

$$G_{\mathrm{I}} = \frac{4(a_1 + a_2)C_{11}\pi\sigma^{0^2}_{21}}{(1-f)\left(C^2_{11} - C^2_{12}\right)}, \quad (28)$$

and for Mode II:

$$G_{\mathrm{II}} = \frac{\left\{(4a_1 + a_2)(C_{11} + C_{12})C^2_{11} - [8a_1C_{11} + a_2(C_{11} + C_{12})]C^2_{12}\right\}\pi\sigma^{0^2}_{22}}{(1-f)\left(C^2_{11} - C^2_{12}\right)\left[(C_{11} + C_{12})C_{11} - 2C^2_{12}\right]}, \quad (29)$$

and for Mode III:

$$G_{\mathrm{III}} = \frac{2(a_1 + a_2)\pi\sigma^{0^2}_{23}}{(1-f)C_{44}}. \quad (30)$$

Equations 28–30 are the closed forms of the energy release rate for N elliptical flaws embedded in an infinite solid under distinct types of mechanical loading. These forms are a function of the aspect ratio and the volume fraction of flaws, the type of the loading, and the material properties.

As an illustrated example to emphasize the physical dimension of these closed forms for the energy release rate, elliptical flaws in an iron are considered. The iron is a cubic crystal material with the following material properties:

$$\begin{aligned} C_{11} &= C_{22} = C_{33} = 242\,\mathrm{Gpa}, \\ C_{44} &= C_{55} = C_{66} = 146.5\,\mathrm{Gpa}, \\ C_{12} &= C_{13} = C_{23} = 112\,\mathrm{Gpa}, \end{aligned} \quad (31)$$

Fig. 1 depicts the numerical demonstration for the closed forms of the energy release rates in Equations 28–30, where G_{I}, G_{II}, and G_{III} are found to increase with the volume fraction f of the flaws when the aspect ratio of the flaw $a_1/a_2 = 100$. Here G_{I} and G_{II} are almost the same and larger than G_{III} as the f increases. Fig. 2 exhibits that the energy release rates linearly increase with respect to the extension of the aspect ratio of the elliptical flaw at $f = 2\%$. These results reveal that the flaws are more difficult to be ruptured in Mode III than other modes as the volume fraction or the aspect ratio of the flaws increases.

5. Critical failure stresses

The critical stress for N flaws to be distended under distinct mechanical loading can be determined according to the Griffith [1] criterion:

$$\frac{\partial}{\partial a_1}\sum_{i=1}^{N}(\Delta W + 2\pi a_1 a_2 \gamma)_i = 0, \quad (32)$$

where γ denotes the surface energy density of the flawed material. Herein, the value of γ is selected as 1 Gpa for numerical simulation. Substituting ε^*_{ji} given by Equations 17–21 into condition 25 leads to critical stresses

$$\sigma^c_{21} = \sqrt{\frac{(1-f)\gamma a_2\left(C^2_{11} - C^2_{12}\right)}{2(a_1 + a_2)C_{11}}} \quad (33)$$

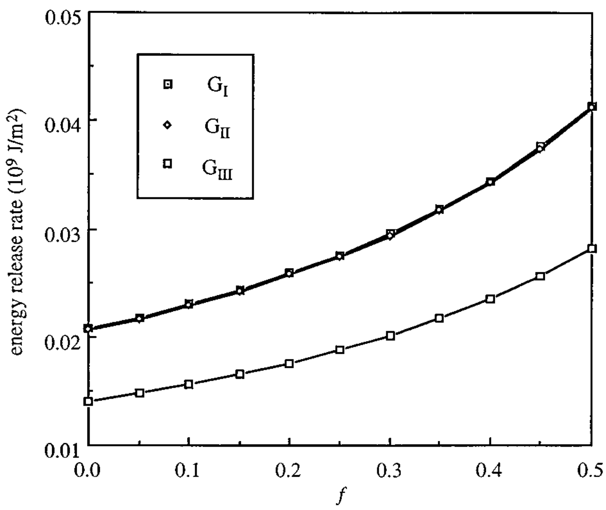

Figure 1 Energy release rates against with f when $a_1/a_2 = 100$.

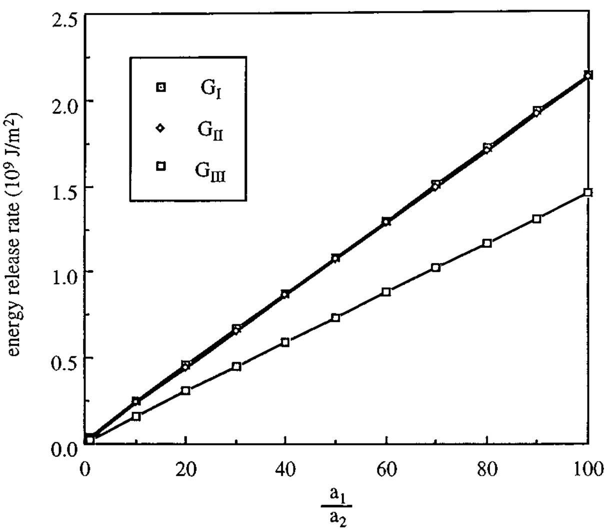

Figure 2 Energy release rates against a_1/a_2 when $f = 2\%$.

for Mode I, and

$$\sigma_{22}^{c} = \sqrt{\frac{2(1-f)\gamma a_2\left(C_{11}^2 - C_{12}^2\right)\left[(C_{11}+C_{12})C_{11} - 2C_{12}^2\right]}{[8a_1C_{11} + a_2(C_{11}+C_{12})]\left[C_{12}^2 - 4(a_1+a_2)(C_{11}+C_{12})C_{11}^2\right]}} \tag{34}$$

for Mode II, and

$$\sigma_{23}^{c} = \sqrt{\frac{2(1-f)\gamma a_2 C_{44}}{2a_1 + a_2}} \tag{35}$$

for Mode III.

The numerical demonstration with the iron, whose elastic constants are given in Equation 31, for the closed forms of the critical stresses in Equations 33–35 is illustrated in Figs 3–4. As clearly shown in these figures, the critical stresses are strongly dependent on the aspect ratio a_1/a_2 and the volume fraction f of the flaws.

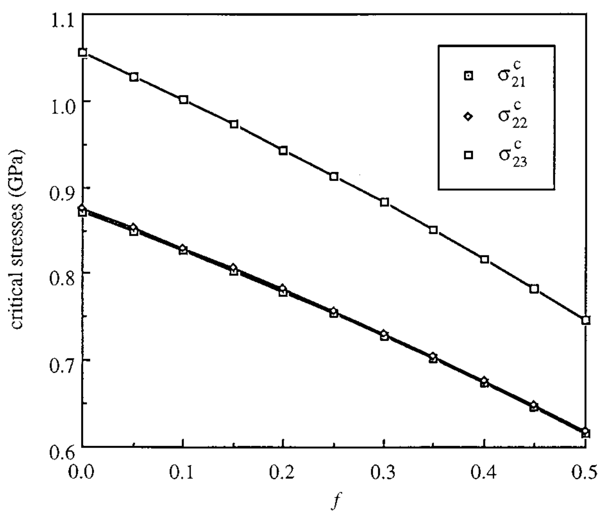

Figure 3 Critical stresses against f when $a_1/a_2 = 100$.

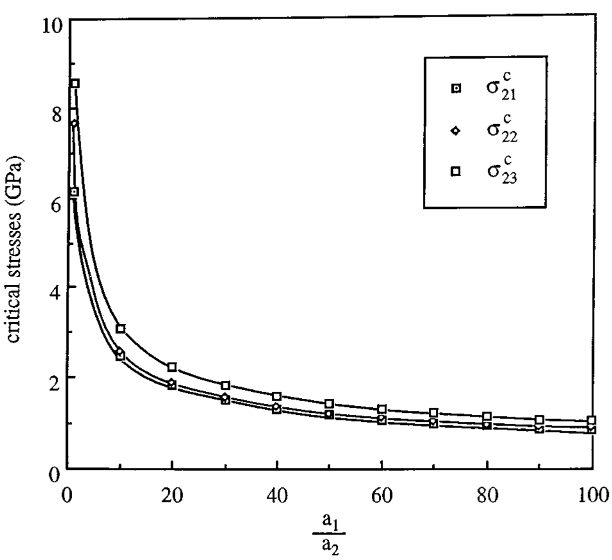

Figure 4 Critical stresses against a_1/a_2 when $f = 2\%$.

Fig. 3 displays the decreasing critical stresses as volume fraction f increases at $a_1/a_2 = 100$. Fig. 4 shows that the critical stresses are monotonously decreased with respect to the extension of aspect ratio of the flaws at $f = 2\%$. Nevertheless, σ_{21}^{c} and σ_{22}^{c} are nearly equal, and the values of σ_{23}^{c} are always less than those of σ_{21}^{c} and σ_{22}^{c} as the range of the aspect ratio or the volume fraction of flaws increases. These indicate that the flaws are easier to be ruptured in Mode I and Mode II than Mode III as the flaw volume fraction or the flaw aspect ratio increases.

6. Summary

This study presents the fracture criterion in a closed form for an infinite anisotropic solid containing multiply elliptical flaws separately subjected to three modes of applied loading. The energy release rates are introduced to quantitatively determine the flaws' extension force. In addition, the critical stresses are employed to forecast the trade of the flaw propagation. The closed forms for energy release rate and critical stresses indicate that they are functions of the aspect ratio and the volume fraction of the flaws, the modes of the loading, and the material properties. According to our results for an iron, energy release rates increase with the volume fraction of flaws, in which G_{III} is less than G_{I} and G_{II} as the volume fraction f of flaws increases; critical stresses decrease as f increases in which σ_{23}^{c} is larger than σ_{21}^{c} and σ_{22}^{c}. Numerical demonstration also illustrates that the energy release rates linearly increase with respect to the extension of the aspect ratio of flaws at a fixed volume fraction, while critical stresses are monotonously decreased. These reveal that the flaws in an iron are more difficult to be ruptured in Mode III than other modes as the volume fraction or the aspect ratio of flaw increases. Finally, it is noted that the formulation presented in the present paper is applicable not only to the flaws in a cubic crystal solid but also to any elastic anisotropic elliptical flaw problems.

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